Math 246A Lecture 28 Notes

Daniel Raban

December 3, 2018

1 Harnack's Principle and Subharmonic Functions

1.1 Harnack's principle

Theorem 1.1 (Harnack's principle)**.** *Let Ω_n be a domain for $n = 1, 2, \dots$, and let Ω be a domain. Let $u_n : \Omega_n \to \mathbb{R}$ be harmonic. Suppose that:*

1. *For all compact $K \subseteq \Omega$, there exists n_K such that if $n \geq n_K$, then $K \subseteq \Omega_n$.*
2. *For all compact $K \subseteq \Omega$, there exists n'_K such that if $n > n'_K$, then $u_{n+1} \geq u_n$ on K.*

Then either

1. *$u_n \to +\infty$ uniformly on K for all compact $K \subseteq \Omega$.*
2. *There exists a harmonic function u on Ω such that $u_n \to u$ uniformly on all compact $K \subseteq \Omega$.*

Proof. Case 1: There exists some $z_0 \in \Omega$ such that $u_n(z_0) \to \infty$. Suppose that $\overline{B(z_0, 2R)} \subseteq \Omega$. Then Harnack's inequality implies that $u_n \to \infty$ uniformly on $\overline{B(z_0, R)}$. Note that $\{z \in \Omega : u_n(z) \to \infty\}$ is nonempty and open. It is also closed. By the connectedness of Ω, $\Omega = \{z \in \Omega : u_n(z) \to \infty\}$. This is the first conclusion.

Case 2: For all $z_0 \in \Omega$, $\sup_n u_n(z_0) < \infty$. Then for compact $K \subseteq \Omega$, $K \subseteq \bigcup_{j=1}^N B(z_j, R_j)$, where $\overline{B(z_j, 2R_j)} \subseteq \Omega$. Then there exists some N_1 such that if $n \geq N_1$, then $u_n \leq M$ on $\tilde{K} = \bigcup_{i=1}^N \overline{B(z_j, (3/2)R_j)} \supseteq K$ by Harnack's inequality. On $B(z_j, 2R_j)$, $u_n = \operatorname{Re}(F_n)$ where $\operatorname{Im}(F_n)(z_j) = 0$, and F_n is holomorphic. Let $f_n = e^{F_n}$; then $|f_n| \leq e^M$, and $|f_n| \geq e^{-M}$. There is a subsequence f_{n_j} converging to some holomorphic g on compact sets, so $u_n \to \log|g|$, where $g \neq 0$. So $u_{n_j} \to u$ on K. $\square$

1.2 Subharmonic functions

We start with an analogy.

Definition 1.1. Suppose $u : (a, b) \to \mathbb{R}$ is C^2. We say that u is **linear** if $u(z) = \alpha z + \beta$.

Then u is linear iff $u(ta+(1-t)b) = tu(a)+(1-r)u+b$. This is if and only if $u_{xx} = 0$.

Definition 1.2. A function u is **convex** if $u(ta+(1-t)b) \leq tu(a)+(1-t)u(b)$.

If $u \in C^2$, then this is equivalent to $u_{xx} \geq 0$.

There is some merit in relaxing an equality to get an inequality. Subharmonic functions will be the same situation but with harmonic instead of linear.

Definition 1.3. Let Ω be a domain and let $u : \Omega \to \mathbb{R}$ be continuous. Then u is **subharmonic** if when the following conditions hold:

1. $\overline{B(z_0, R)} \subseteq \Omega$,
2. v is continuous on $\overline{B(z_0, R)}$,
3. v is harmonic on $B(z_0, R)$,
4. $u \leq v$ on $\partial B(z_0, R)$,

then $u \leq v$ on $B(z_0, R)$.

Example 1.1. If u is harmonic, then u is subharmonic.

Example 1.2. Let $u \in C^2$, and let $u_{xx} + u_{yy} \geq 0$. Then u is subharmonic. Suppose $u - v$ has a local max at $z_0 \in \Omega$, where v is harmonic. It is an exercise in calculus to show that $u_{xx} + u_{yy} < 0$. Look at the Taylor expansion around z_0.

Theorem 1.2. *Suppose u is continuous on Ω. Then v is subharmonic if and only if for all $z_0 \in \Omega$, there exists $R(z_0) > 0$ such that for $0 < R < R(z_0)$,*

$$u(z_0) \leq \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + Re^{i\theta})\, d\theta.$$

Equality here is the mean value property for harmonic functions. The proof is just the maximum principle.

Example 1.3. If u is subharmonic and $\lambda > 0$, then λu is subharmonic.

Example 1.4. If u_1, u_2 are subharmonic, then $u_1 + u_2$ is subharmonic.

Example 1.5. If u_1, u_2 are subharmonic, then $\max(u_1, u_2)$ is subharmonic.

Example 1.6 (regularization)**.** Let $\overline{B(z_0, R)} \subseteq \Omega$, let u be subharmonic in Ω, let $P_z = \frac{R^2 - |z-z_0|^2}{|Re^{i\theta} - (z-z_0)|^2}$, and let

$$U(z) = \begin{cases} u(z) & z \in \Omega \setminus B(z_0, R) \\ \frac{1}{2\pi} \int_{\partial B} P_z(\theta) u(z_0 + Re^{i\theta})\, d\theta & \text{on B} \end{cases}$$

Then U is continuous, subharmonic, $U \geq u$, and U is harmonic on B.

Example 1.7. Let $f \in H(\Omega)$ not be identically zero. Extend the definition of subharmonic to the case where $u : \Omega \to [-\infty, \infty]$ is upper semicontinuous. Then $\log |f(z)|$ is subharmonic because

$$\log |f(z)| \leq \frac{1}{2\pi} \int_0^{2\pi} \log |f(z_0 + Re^{i\theta})| \, d\theta,$$

where $\overline{B(z_0, R)} \subseteq \Omega$. This is **Jensen's inequality**.